

# DETERMINANTS ASSOCIATED TO TRACES ON OPERATOR BIMODULES

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**ABSTRACT.** Given a  $\text{II}_1$ -factor  $\mathcal{M}$  with tracial state  $\tau$  and given an  $\mathcal{M}$ -bimodule  $\mathcal{E}(\mathcal{M}, \tau)$  of operators affiliated to  $\mathcal{M}$  we show that traces on  $\mathcal{E}(\mathcal{M}, \tau)$  (namely, linear functionals that are invariant under unitary conjugation) are in bijective correspondence with rearrangement-invariant linear functionals on the corresponding symmetric function space  $E$ . We also show that, given a positive trace  $\varphi$  on  $\mathcal{E}(\mathcal{M}, \tau)$ , the map  $\det_\varphi : \mathcal{E}_{\log}(\mathcal{M}, \tau) \rightarrow [0, \infty)$  defined by  $\det_\varphi(T) = \exp(\varphi(\log |T|))$  when  $\log |T| \in \mathcal{E}(\mathcal{M}, \tau)$  and 0 otherwise, is multiplicative on the  $*$ -algebra  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$  that consists of all affiliated operators  $T$  such that  $\log_+(|T|) \in \mathcal{E}(\mathcal{M}, \tau)$ . Finally, we show that all multiplicative maps on the invertible elements of  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$  arise in this fashion.

## 1. INTRODUCTION

Let  $\mathcal{M}$  be a von Neumann algebra factor of type  $\text{II}_1$ , with tracial state  $\tau$ . Assume  $\mathcal{M}$  has separable predual. The Fuglede–Kadison determinant [8], is the multiplicative map  $\Delta_\tau : \mathcal{M} \rightarrow [0, \infty)$  defined by

$$\Delta_\tau(T) = \lim_{\epsilon \rightarrow 0^+} \exp(\tau(\log(|T| + \epsilon))). \quad (1)$$

In this paper, we prove multiplicativity of analogous determinants corresponding to arbitrary positive traces on arbitrary  $\mathcal{M}$ -bimodules of affiliated operators.

Choose any normal representation of  $\mathcal{M}$  on a Hilbert space and let  $\mathcal{S}(\mathcal{M}, \tau)$  be the  $*$ -algebra of (possibly unbounded) operators on the Hilbert space affiliated to  $\mathcal{M}$ . This algebra, often called the Murray-von Neumann algebra of  $\mathcal{M}$ , is independent of the representation. See, for example, Section 6 of [11] for an exposition of this theory. Let  $\text{Proj}(\mathcal{M})$  denote the set of projections (i.e., self-adjoint idempotents) in  $\mathcal{M}$ . For  $A \in \mathcal{S}(\mathcal{M}, \tau)$  and  $t \in (0, 1)$ ,  $\mu(t, A)$  denotes the generalized singular number of  $A$ , defined by

$$\mu(t, A) = \inf\{\|A(1 - p)\| \mid p \in \text{Proj}(\mathcal{M}), \tau(p) \leq t\},$$

where  $\|\cdot\|$  is the operator norm. This goes back to Murray and von Neumann; see, for example, Section 2.3 of [14] for some basic theory. We will write simply  $\mu(A)$  for the function  $t \mapsto \mu(t, A)$ , which is nonincreasing and right continuous.

Let  $E$  be a complex vector space of measurable functions on  $[0, 1]$  with the property that if  $f$  and  $g$  are measurable functions with  $f^* \leq g^*$  and  $g \in E$ , then  $f \in E$ , where  $f^*$  denotes the decreasing rearrangement of  $|f|$ . Following [14], we will call such a space  $E$  a Calkin function space. Note that  $f \in E$  implies that the dilation  $D_2 f$  lies in  $E$ , where  $D_2 f(t) = f(t/2)$ . In particular, every nonzero Calkin function space contains  $L_\infty[0, 1]$ . The corresponding  $\mathcal{M}$ -bimodule  $\mathcal{E}(\mathcal{M}, \tau)$  is the set of all  $A \in \mathcal{S}(\mathcal{M}, \tau)$  such that  $\mu(A) \in E$ .

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This correspondence, sometimes called the Calkin correspondence in the setting of  $(\mathcal{M}, \tau)$ , is a bijection from the set of all Calkin function spaces onto the set of all operator  $\mathcal{M}$ -bimodules, by which we mean subspaces of  $\mathcal{S}(\mathcal{M}, \tau)$  that are closed under left and right multiplication by elements of  $\mathcal{M}$ , and it goes back to Guido and Isola [9]. See Theorem 2.4.4 of [14] for the formulation used here. An equivalent version of this is also described in [4]. Note that if  $\mathcal{A} \subseteq \mathcal{M}$  is any unital abelian von Neumann subalgebra that is diffuse (i.e., has no minimal projections), then the  $*$ -algebra  $\mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}})$  of affiliated operators is naturally embedded in  $\mathcal{S}(\mathcal{M}, \tau)$  and, upon identifying  $\mathcal{A}$  with  $L_\infty(0, 1)$ , the elements of  $\mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}})$  are naturally identified with measurable functions on  $(0, 1)$ . Under these identifications, we have  $E = \mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$ .

By a trace on  $\mathcal{E}(\mathcal{M}, \tau)$ , we mean a linear functional  $\varphi$  of  $\mathcal{E}(\mathcal{M}, \tau)$  such that  $\varphi(UAU^*) = \varphi(A)$  for every  $A \in \mathcal{E}(\mathcal{M}, \tau)$  and every unitary  $U \in \mathcal{M}$ . A functional  $\varphi_0$  of  $E$  is said to be rearrangement-invariant if  $\varphi_0(f) = \varphi_0(g)$  whenever  $f, g \in E$ ,  $f, g \geq 0$  and  $f^* = g^*$ .

The difficult half of the following result is essentially proved in [13]. The proof of the other half is similar to the proof of Lemma 9.4 of [6].

**Theorem 1.1.** *Let  $\mathcal{M}$  be a  $II_1$ -factor with separable predual. Let  $E$  be a Calkin function space and let  $\mathcal{E}(\mathcal{M}, \tau)$  be the corresponding  $\mathcal{M}$ -bimodule. There is a bijection from the set of all traces of  $\mathcal{E}(\mathcal{M}, \tau)$  onto the set of all rearrangement-invariant functionals of  $E$ , whereby a trace  $\varphi$  of  $\mathcal{E}(\mathcal{M}, \tau)$  is mapped to a functional  $\varphi_0$  of  $E$  satisfying*

$$\varphi_0(\mu(A)) = \varphi(A) \text{ whenever } A \in \mathcal{E}(\mathcal{M}, \tau) \text{ and } A \geq 0. \quad (2)$$

*Proof.* Suppose  $\varphi_0 : E \rightarrow \mathbb{C}$  is a rearrangement-invariant linear functional. By the proof of (part of) Theorem 5.2 of [13], there is a trace  $\varphi : \mathcal{E}(\mathcal{M}, \tau) \rightarrow \mathbb{C}$  satisfying (2). The statement of that theorem includes additional assumptions about  $E$ , namely, that it carries a rearrangement-invariant complete norm. However, the proof found in [13] is valid, verbatim, in the more general situation considered here.

Suppose  $\varphi : \mathcal{E}(\mathcal{M}, \tau) \rightarrow \mathbb{C}$  is a trace. We will now show that for any  $A \in \mathcal{E}(\mathcal{M}, \tau)$  that is positive,  $\varphi(A)$  depends only on  $\mu(A)$ . Indeed, let  $A_1, A_2 \in \mathcal{E}(\mathcal{M}, \tau)$  be such that  $A_1, A_2 \geq 0$  and  $\mu(A_1) = \mu(A_2)$ . Set

$$B_k = \sum_{n \geq 0} n 1_{[n, n+1)}(A_k), \quad C_k = A_k - B_k, \quad k = 1, 2.$$

Clearly, positive operators  $B_1$  and  $B_2$  have discrete spectrum and  $\mu(B_1) = \mu(B_2)$ . Since  $\mathcal{M}$  is a factor, one can choose a unitary element  $U \in \mathcal{M}$  such that  $B_1 = UB_2U^{-1}$ . Clearly,  $\varphi(B_1) = \varphi(UB_2U^{-1}) = \varphi(B_2)$ . By Theorem 2.3 in [7], we have  $\varphi|_{\mathcal{M}} = c_\varphi \tau|_{\mathcal{M}}$  for a constant  $c_\varphi$ . For bounded positive operators  $C_1$  and  $C_2$ , we have  $\mu(C_1) = \mu(C_2)$  and also, therefore,

$$\varphi(C_1) = c_\varphi \tau(C_1) = c_\varphi \tau(C_2) = \varphi(C_2).$$

Thus, we get

$$\varphi(A_1) = \varphi(B_1) + \varphi(C_1) = \varphi(B_2) + \varphi(C_2) = \varphi(A_2).$$

Let  $\mathcal{A}$  be any unital, diffuse, abelian von Neumann subalgebra of  $\mathcal{M}$ . As described above,  $E$  is naturally identified with  $\mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$ , and restricting  $\varphi$  to this subalgebra yields a linear functional  $\varphi_0$  on  $E$ , which is rearrangement-invariant and satisfies (2), because of the fact that  $\varphi(A)$  depends only on  $\mu(A)$  for all  $A \geq 0$ . Using (2), we see that the functional  $\varphi_0$  does not depend on  $\mathcal{A}$ , namely, does not depend on which copy of  $E$  we chose in  $\mathcal{E}(\mathcal{M}, \tau)$ .

Finally, as  $\varphi$  is uniquely determined by  $\varphi_0$  and the condition (2), we see that the map  $\varphi \mapsto \varphi_0$  is the desired bijection.  $\square$

For convenience, we will use also  $\varphi$ , instead of  $\varphi_0$ , to denote the functional on  $E$  corresponding to a trace  $\varphi$  on  $\mathcal{E}(\mathcal{M}, \tau)$ .

For example, taking  $E$  to be the function space  $L_1$  of complex-valued functions on  $[0, 1]$  that are integrable with respect to Lebesgue measure, the corresponding bimodule is  $\mathcal{L}_1(\mathcal{M}, \tau)$ . Moreover, the functional  $f \mapsto \int_0^1 f(t) dt$  on  $L_1$  corresponds to the usual trace  $\tau$  on  $\mathcal{L}_1(\mathcal{M}, \tau)$ . Other examples of traces on bimodules are provided by the Dixmier traces on Marcinkiewicz bimodules, which are of interest in noncommutative geometry. See, for example, [3], [2] and [12]; particularly, consider the treatment of functionals supported at zero, but adapted to the case of a  $\text{II}_1$ -factor  $\mathcal{M}$ , namely, corresponding to function spaces on  $[0, 1]$ . A specific case (essentially, taken from [3]) is found in Example 3.3.

The Fuglede-Kadison determinant mentioned at the start of this introduction is actually naturally defined on the space, sometimes denoted  $\mathcal{L}_{\log}(\mathcal{M}, \tau)$ , of all  $T \in \mathcal{S}(\mathcal{M}, \tau)$  such that  $\log_+(|T|) \in \mathcal{L}_1(\mathcal{M}, \tau)$ , where  $\log_+(t) = \max(\log(t), 0)$ . See [10] for a development of  $\Delta_\tau$  in this generality, including a proof of multiplicativity.

In the rest of this paper, we will for the most part consider only *positive* traces  $\varphi$ , namely, those satisfying

$$A \geq 0 \implies \varphi(A) \geq 0$$

(the exception being Lemma 2.8). Positive traces correspond, under the rubrik of Theorem 1.1, to positive rearrangement-invariant linear functionals. In the following, we use the function  $\log_-(t) = -\min(\log(t), 0)$ ; thus,  $\log = \log_+ - \log_-$ .

**Definition 1.2.** Let  $\mathcal{M}$  be a  $\text{II}_1$ -factor and consider a positive trace  $\varphi$  on an  $\mathcal{M}$ -bimodule  $\mathcal{E}(\mathcal{M}, \tau)$ . Let  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$  be the set of all  $T \in \mathcal{S}(\mathcal{M}, \tau)$  such that  $\log_+(|T|) \in \mathcal{E}(\mathcal{M}, \tau)$  and for such  $T$  let

$$\det_\varphi(T) = \begin{cases} \exp(\varphi(\log(|T|))), & \ker T = \{0\} \text{ and } \log_-(|T|) \in E \\ 0, & \ker T = \{0\} \text{ and } \log_-(|T|) \notin E \\ 0, & \ker T \neq \{0\}. \end{cases}$$

Thus, in the case  $E = L_1$  and  $\varphi = \tau$ , we have the Fuglede-Kadison determinant:  $\det_\tau = \Delta_\tau$ . The natural domain of this determinant by the above rubric should be written  $\mathcal{L}_{1, \log}(\mathcal{M}, \tau)$ , but we will write  $\mathcal{L}_{\log}(\mathcal{M}, \tau)$  for this, in keeping with earlier convention (cf [5], [6]).

The main result of this paper is:

**Theorem 1.3.** *For an arbitrary Calkin function space  $E$  on  $[0, 1]$  and arbitrary positive trace  $\varphi$  on the corresponding bimodule  $\mathcal{E}(\mathcal{M}, \tau)$ , the set  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$  is a  $*$ -subalgebra of  $\mathcal{S}(\mathcal{M}, \tau)$  and, if  $A, B \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$ , then*

$$\det_\varphi(AB) = \det_\varphi(A)\det_\varphi(B). \quad (3)$$

The proof, presented in the next section, relies on Fuglede and Kadison's result [8] that  $\Delta_\tau$  is multiplicative on  $\mathcal{M}$  and on the characterization from [4] of sums of  $(\mathcal{E}(\mathcal{M}, \tau), \mathcal{M})$ -commutators. Thus, a special case of this proof yields an alternative proof of Haagerup and Schultz's result [10] about the extension of the Fuglede-Kadison determinant to  $\mathcal{L}_{\log}(\mathcal{M}, \tau)$ .

**Remark 1.4.** It is immediate that  $\det_\varphi(1) = 1$  and, for  $T \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$ ,  $\det_\varphi(T) = 0$  if and only if  $T$  fails to be invertible in  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ .

**Remark 1.5.** In the case that  $\varphi = 0$ , we clearly have, for  $T \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$ ,

$$\det_\varphi(T) = \begin{cases} 1 & \text{if } T \text{ is invertible in } \mathcal{E}_{\log}(\mathcal{M}, \tau) \\ 0 & \text{otherwise.} \end{cases}$$

However, if  $\varphi \neq 0$ , then  $\det_\varphi$  is onto  $[0, \infty)$ .

**Remark 1.6.** It is not difficult to see, in the case  $\varphi = \tau$ , that Definition 1.2 agrees with the definition by equation (1), in fact even for all  $T \in \mathcal{L}_{\log}(\mathcal{M}, \tau)$ . However, the analogous statement is not true for general traces  $\varphi$ . In fact, it obviously fails when  $\varphi = 0$ , (see Remark 1.5, above). See Example 3.3 for specific examples of this failure when  $\varphi \neq 0$ .

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**Proposition 1.7.** *For an arbitrary Calkin function space  $E$  on  $[0, 1]$  and an arbitrary map*

$$m : \mathcal{E}_{\log}(\mathcal{M}, \tau) \rightarrow [0, \infty)$$

*that is multiplicative, order-preserving and nonzero, there exists a positive trace  $\varphi$  on  $\mathcal{E}(\mathcal{M}, \tau)$  such that  $m(X) = \det_\varphi(X)$  for every invertible element  $X$  in  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ .*

We will show (in Proposition 3.2) that we cannot hope for  $m$  to agree with  $\det_\varphi$  on all of  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ .

The proofs of Theorem 1.3 and Proposition 3.2 are contained in the next two sections.

## 2. PROOF OF THEOREM 1.3

Let us begin by describing some further notation and standard conventions.

- $S(0, 1)$  will denote the set of all complex-valued Borel measurable functions on  $[0, 1]$  and  $L_\infty$  will denote the set of all essentially bounded elements of  $S(0, 1)$ . As usual, we consider functions that are equal almost everywhere to be the same.
- We will apply the Borel functional calculus to self-adjoint elements  $T \in \mathcal{S}(\mathcal{M}, \tau)$ , and will also use the standard notation  $T_+ = \max(T, 0)$  and  $T_- = -\min(T, 0)$ .
- For self-adjoint  $A \in \mathcal{S}(\mathcal{M}, \tau)$ , we consider its eigenvalue function (or spectral scale), defined for  $t \in (0, 1)$  by

$$\lambda(t, A) = \inf\{s \in \mathbb{R} \mid \tau(1_{(s, \infty)}(A)) \leq t\},$$

where, in accordance with notation for the Borel functional calculus,  $1_{(s, \infty)}(A)$  denotes the spectral projection of  $A$  associated to the interval  $(s, \infty)$ . This also goes back to Murray and von Neumann. We will write simply  $\lambda(A)$  for the function  $t \mapsto \lambda(t, A)$ , which is nonincreasing and right continuous. Note that, if  $A \geq 0$ , then  $\lambda(A) = \mu(A)$ . Moreover, when  $a \leq b$ , with  $a \leq \lim_{t \rightarrow 0} \lambda(t, A)$  and  $b \geq \lim_{t \rightarrow 1} \lambda(t, A)$ , we have

$$\tau(A 1_{[a, b]}(A)) = \int_c^d \lambda(t, A) dt, \quad (4)$$

$$\tau(1_{[a, b]}(A)) = d - c, \quad (5)$$

where

$$c = \inf\{s \mid \lambda(s, A) \leq b\}, \quad d = \sup\{s \mid \lambda(s, A) \geq a\}.$$

For any  $T \in \mathcal{S}(\mathcal{M}, \tau)$ , since  $\mu(T) = \mu(|T|) = \lambda(|T|)$ , from (5), we get

$$\tau(1_{[0, \mu(t, T)]}(|T|)) \geq 1 - t. \quad (6)$$

- The following inequalities are standard (see, for example, Corollary 2.3.16 of [14]): for all  $A, B \in \mathcal{S}(\mathcal{M}, \tau)$ , if  $s, t > 0$  and  $s + t < 1$ , then

$$\mu(s + t, A + B) \leq \mu(s, A) + \mu(t, B), \quad (7)$$

$$\mu(s + t, AB) \leq \mu(s, A)\mu(t, B). \quad (8)$$

- If a function  $f$  on  $(0, 1)$  is right-continuous and monotone, then we will let  $\tilde{f}$  denote left-continuous version, namely,

$$\tilde{f}(x) = \lim_{t \rightarrow x^-} f(t). \quad (9)$$

**Lemma 2.1.** *Let  $T, S \in \mathcal{S}(\mathcal{M}, \tau)$  be self-adjoint. Then for every  $t \in (0, \frac{1}{4})$ , we have*

$$\left| \int_{2t}^{1-2t} (\log(\mu(u, e^T e^S)) - \lambda(u, T) - \lambda(u, S)) du \right| \leq 8t(\mu(t, T) + \mu(t, S)).$$

*Proof.* Fix  $t \in (0, \frac{1}{4})$  and, using the continuous functional calculus, set

$$\begin{aligned} T_0 &= \min\{T_+, \mu(t, T)\} - \min\{T_-, \mu(t, T)\}, \\ S_0 &= \min\{S_+, \mu(t, S)\} - \min\{S_-, \mu(t, S)\}. \end{aligned}$$

We have

$$T - T_0 = (T_+ - \mu(t, T))_+ - (T_- - \mu(t, T))_+,$$

$$|T - T_0| = (T_+ - \mu(t, T))_+ + (T_- - \mu(t, T))_+ = (|T| - \mu(t, T))_+.$$

Thus, we have  $(T - T_0)1_{[0, \mu(t, T)]}(|T|) = 0$  and, using (6), we get  $\mu(t, T - T_0) = 0$ ; similarly, we have  $\mu(t, S - S_0) = 0$ . Using (8), for every  $u \in (2t, 1)$  we have

$$\begin{aligned} \mu(u, e^T e^S) &= \mu(u, e^{T-T_0} \cdot e^{T_0} e^{S_0} \cdot e^{S-S_0}) \leq \mu(t, e^{T-T_0}) \mu(u-2t, e^{T_0} e^{S_0}) \mu(t, e^{S-S_0}), \\ \mu(u, e^{T_0} e^{S_0}) &= \mu(u, e^{T_0-T} \cdot e^T e^S \cdot e^{S_0-S}) \leq \mu(t, e^{T_0-T}) \mu(u-2t, e^T e^S) \mu(t, e^{S_0-S}), \end{aligned}$$

Since  $\mu(t, e^{T-T_0}) \leq 1$  and  $\mu(t, e^{T_0-T}) \leq 1$  and similarly for  $S - S_0$ , we get

$$\mu(u, e^T e^S) \leq \mu(u-2t, e^{T_0} e^{S_0}), \quad \mu(u, e^{T_0} e^{S_0}) \leq \mu(u-2t, e^T e^S).$$

Thus, for  $u \in (2t, 1-2t)$ , we have

$$\mu(u+2t, e^{T_0} e^{S_0}) \leq \mu(u, e^T e^S) \leq \mu(u-2t, e^{T_0} e^{S_0}).$$

It follows that

$$\int_{4t}^1 \log(\mu(u, e^{T_0} e^{S_0})) du \leq \int_{2t}^{1-2t} \log(\mu(u, e^T e^S)) du \leq \int_0^{1-4t} \log(\mu(u, e^{T_0} e^{S_0})) du. \quad (10)$$

Since  $-\mu(t, T) \leq T_0 \leq \mu(t, T)$  and similarly for  $S_0$ , we also have

$$e^{-\mu(t, T) - \mu(t, S)} \leq \mu(e^{T_0} e^{S_0}) \leq e^{\mu(t, T) + \mu(t, S)}.$$

Thus,

$$\|\log(\mu(e^{T_0} e^{S_0}))\|_\infty \leq \mu(t, T) + \mu(t, S).$$

In particular,

$$\begin{aligned} \left| \int_0^{4t} \log(\mu(u, e^{T_0} e^{S_0})) du \right| &\leq 4t \|\log(\mu(e^{T_0} e^{S_0}))\|_\infty \leq 4t(\mu(t, T) + \mu(t, S)), \\ \left| \int_{1-4t}^1 \log(\mu(u, e^{T_0} e^{S_0})) du \right| &\leq 4t \|\log(\mu(e^{T_0} e^{S_0}))\|_\infty \leq 4t(\mu(t, T) + \mu(t, S)). \end{aligned}$$

Using (10), we get

$$\left| \int_{2t}^{1-2t} \log(\mu(u, e^T e^S)) du - \int_0^1 \log(\mu(u, e^{T_0} e^{S_0})) du \right| \leq 4t(\mu(t, T) + \mu(t, S)).$$

Since the Fuglede-Kadison determinant  $\Delta_\tau$  is multiplicative on  $\mathcal{M}$ , we have

$$\begin{aligned} \int_0^1 \log(\mu(u, e^{T_0} e^{S_0})) du &= \log(\Delta_\tau(e^{T_0} e^{S_0})) \\ &= \log(\Delta_\tau(e^{T_0})) + \log(\Delta_\tau(e^{S_0})) = \tau(T_0) + \tau(S_0). \end{aligned}$$

But using

$$\left| \tau(T_0) - \int_{2t}^{1-2t} \lambda(u, T) du \right| \leq 4t\mu(t, T),$$

and the same also for  $S$ , the assertion follows.  $\square$

In the following, we use the notation (9) for the left-continuous versions of monotone functions. (Though, as elements of  $E$ ,  $\mu(T)$  and the left-continuous version  $\tilde{\mu}(T)$  are identified, these functions  $\mu(T)$  and similarly  $\lambda(T)$  are of interest aside from their membership in  $E$ , and for correctness at all points of  $(0, 1)$  we must use their left-continuous versions in the following inequalities and elsewhere below.)

**Lemma 2.2.** *If  $S, T \in \mathcal{S}(\mathcal{M}, \tau)$  are self-adjoint, then for all  $u \in (0, 1)$ , we have*

$$-\tilde{\mu}\left(\frac{1-u}{2}, T\right) - \tilde{\mu}\left(\frac{1-u}{2}, S\right) \leq \log(\mu(u, e^T e^S)) \leq \mu\left(\frac{u}{2}, T\right) + \mu\left(\frac{u}{2}, S\right). \quad (11)$$

*Proof.* Using (8), we get

$$\begin{aligned} \mu(u, e^T e^S) &\leq \mu\left(\frac{u}{2}, e^T\right) \mu\left(\frac{u}{2}, e^S\right) \leq \mu\left(\frac{u}{2}, e^{T+}\right) \mu\left(\frac{u}{2}, e^{S+}\right) = e^{\mu(\frac{u}{2}, T_+) + \mu(\frac{u}{2}, S_+)} \\ &\leq e^{\mu(\frac{u}{2}, T) + \mu(\frac{u}{2}, S)}, \end{aligned} \quad (12)$$

which yields the right-most inequality in (11). Replacing  $S$  with  $-T$  and  $T$  with  $-S$  in (12), we get

$$\mu(u, e^{-S} e^{-T}) \leq e^{\mu(\frac{u}{2}, T_-) + \mu(\frac{u}{2}, S_-)}, \quad \tilde{\mu}(u, e^{-S} e^{-T}) \leq e^{\tilde{\mu}(\frac{u}{2}, T_-) + \tilde{\mu}(\frac{u}{2}, S_-)}. \quad (13)$$

As is well known and easy to show,

$$\mu(u, e^T e^S) = \frac{1}{\tilde{\mu}(1-u, e^{-S} e^{-T})}.$$

Thus, replacing  $u$  with  $1-u$  in (13), we get

$$\mu(u, e^T e^S) \geq e^{-\tilde{\mu}(\frac{1-u}{2}, T_-) - \tilde{\mu}(\frac{1-u}{2}, S_-)} \geq e^{-\tilde{\mu}(\frac{1-u}{2}, T) - \tilde{\mu}(\frac{1-u}{2}, S)},$$

which yields the left-most inequality in (11).  $\square$

The next lemma is a combination of Theorems 3.3.3 and 3.3.4 from [14].

**Lemma 2.3.** *If  $S, T \in \mathcal{M}$  are positive, then*

$$\int_0^t \mu(u, T + S) du \leq \int_0^t (\mu(u, T) + \mu(u, S)) du \leq \int_0^{2t} \mu(u, T + S) du.$$

*Proof.* This follows easily from the fact that, for a positive operator,  $T$ , we have

$$\int_0^t \mu(u, T) du = \sup\{\tau(pT) \mid p \in \text{Proj}(\mathcal{M}), \tau(p) \leq t\}.$$

$\square$

For every function  $f \in S(0, 1)$  that is bounded on compact subsets of  $(0, 1)$ , define

$$(\Psi f)(t) = \begin{cases} \frac{1}{t} \int_t^{1-t} f(s) ds, & 0 < t < \frac{1}{2}, \\ 0, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Clearly,  $\Psi f$  is continuous on  $(0, 1]$  and  $\Psi$  is linear. Note that  $\Psi$  is defined on every function arising as  $\mu(A)$  or  $\lambda(A)$  for  $A \in \mathcal{S}(\mathcal{M}, \tau)$ .

**Lemma 2.4.** *Let  $S, T \in \mathcal{E}(\mathcal{M}, \tau)$  be positive. Then*

$$\Psi(\mu(T + S) - \mu(T) - \mu(S)) \in E.$$

*Proof.* First suppose  $S, T \in \mathcal{M}$  are positive. From Lemma 2.3 and the fact that  $\tau(T) = \int_0^1 \mu(u, T) du$ , we have

$$\int_{2t}^1 \mu(u, T + S) du \leq \int_t^1 (\mu(u, T) + \mu(u, S)) du \leq \int_t^1 \mu(u, T + S) du. \quad (14)$$

For arbitrary positive  $S, T \in \mathcal{S}(\mathcal{M}, \tau)$ , set  $T_n = \min\{T, n\}$  and  $S_n = \min\{S, n\}$ . Since  $\mu(T_n) \uparrow \mu(T)$ ,  $\mu(S_n) \uparrow \mu(S)$  and  $\mu(T_n + S_n) \uparrow \mu(T + S)$ , it follows from the Monotone Convergence Principle that (14) also holds. From (14), we have

$$\left| \int_t^1 (\mu(u, T + S) - \mu(u, T) - \mu(u, S)) du \right| \leq \int_t^{2t} \mu(u, T + S) du \leq t\mu(t, T + S).$$

Thus, for  $t \in (0, \frac{1}{2})$ , we have

$$\begin{aligned} & \left| \int_t^{1-t} (\mu(u, T + S) - \mu(u, T) - \mu(u, S)) du \right| \\ & \leq \left| \int_t^1 (\mu(u, T + S) - \mu(u, T) - \mu(u, S)) du \right| \\ & \quad + \left| \int_{1-t}^1 (\mu(u, T + S) - \mu(u, T) - \mu(u, S)) du \right| \\ & \leq t\mu(t, T + S) + t\mu(1-t, T + S) + t\mu(1-t, T) + t\mu(1-t, S) \leq 4t\mu(t, T + S). \end{aligned}$$

This concludes the proof.  $\square$

**Lemma 2.5.** *Let  $T \in \mathcal{S}(\mathcal{M}, \tau)$  be self-adjoint. Then*

$$\Psi(\lambda(T) - \mu(T_+) + \mu(T_-)) \in L_\infty.$$

*Proof.* If  $T_+ = 0$  or  $T_- = 0$ , then  $\lambda(T) = \mu(T_+) - \mu(T_-)$ . Suppose  $T_+ \neq 0$  and  $T_- \neq 0$ . Let  $t_0$  be the trace of the support projection of  $T_+$ . We have

$$\lambda(u, T) = \begin{cases} \mu(u, T_+), & u \in (0, t_0) \\ -\tilde{\mu}(1-u, T_-), & u \in [t_0, 1). \end{cases}$$

It follows that, for all sufficiently small  $t$ , we have

$$\begin{aligned} t(\Psi\lambda(T))(t) &= \int_t^{t_0} \lambda(u, T) du + \int_{t_0}^{1-t} \lambda(u, T) du \\ &= \int_t^{t_0} \mu(u, T_+) du - \int_{t_0}^{1-t} \mu(1-u, T_-) du = \int_t^{t_0} \mu(u, T_+) du - \int_t^{1-t_0} \mu(u, T_-) du \\ &= \int_t^1 (\mu(u, T_+) - \mu(u, T_-)) du = t(\Psi(\mu(T_+) - \mu(T_-)))(t), \end{aligned}$$

where the last equality holds because the integrand is zero when  $u$  is sufficiently close to 1. Thus,  $\Psi(\lambda(T) - \mu(T_+) + \mu(T_-))(t)$  vanishes for all  $t$  sufficiently small. Since this function is continuous on  $(0, 1]$ , it is bounded.  $\square$

**Lemma 2.6.** *Let  $S, T \in \mathcal{E}(\mathcal{M}, \tau)$  be self-adjoint. Then*

$$\Psi(\lambda(T) + \lambda(S) - \lambda(T + S)) \in E.$$

*Proof.* We have

$$(T + S)_+ - (T + S)_- = T_+ - T_- + S_+ - S_-.$$

Therefore,

$$(T + S)_+ + T_- + S_- = (T + S)_- + T_+ + S_+.$$

Denote the above quantity by  $A$ . From Lemma 2.4, we obtain

$$\Psi(\mu(A) - \mu((T + S)_+) - \mu(T_-) - \mu(S_-)) \in E,$$

$$\Psi(\mu(A) - \mu((T + S)_-) - \mu(T_+) - \mu(S_+)) \in E.$$

Subtracting those formulae, we obtain

$$\Psi(\mu((T + S)_+) - \mu((T + S)_-) - \mu(T_+) + \mu(T_-) - \mu(S_+) + \mu(S_-)) \in E.$$

The assertion follows now from Lemma 2.5 as applied to the operators  $T$ ,  $S$  and  $T + S$ , and the fact that  $E$  contains  $L_\infty$ .  $\square$

In the next result, the notation  $[\mathcal{E}(\mathcal{M}, \tau), \mathcal{M}]$  denotes the space spanned by the set of all commutators of the form  $[S, T] = ST - TS$ , for  $S \in \mathcal{M}$  and  $T \in \mathcal{E}(\mathcal{M}, \tau)$ . It amounts to a reformulation of a special case of Theorem 4.6 of [4].

**Theorem 2.7.** *Let  $T \in \mathcal{E}(\mathcal{M}, \tau)$  be self-adjoint. Then  $T \in [\mathcal{E}(\mathcal{M}, \tau), \mathcal{M}]$  if and only if  $\Psi\lambda(T) \in E$ .*

*Proof.* By Theorem 4.6 of [4],  $T \in [\mathcal{E}(\mathcal{M}, \tau), \mathcal{M}]$  if and only if the function

$$r \mapsto \frac{1}{r} \tau(1_{[0, \mu(r, T)]}(|T|)T)$$

belongs to  $E$ . Thus, it will suffice to show that the function

$$r \mapsto \frac{1}{r} \tau(1_{[0, \mu(r, T)]}(|T|)T) - \Psi\lambda(T)(r) \tag{15}$$

belongs to  $E$ . First suppose  $T_- = 0$ . Then, using  $\lambda(T) = \mu(T)$  and (4), we have

$$\tau(1_{[0, \mu(r, T)]}(|T|)T) = \int_{r'}^1 \mu(t, T) dt,$$

where  $r' = \inf\{s \mid \mu(s, T) \leq \mu(r, T)\}$ . Thus  $r' \leq r$  and, for  $0 < r < \frac{1}{2}$ ,

$$\left| \tau(1_{[0, \mu(r, T)]}(|T|)T) - \int_r^{1-r} \lambda(t, T) dt \right| \leq (r - r')\mu(r, T) + r\mu(1 - r, T) \leq 2r\mu(r, T),$$

which implies that the function (15) belongs to  $E$ .

If  $T_+ = 0$ , then we may of course replace  $T$  by  $-T$  and we are done.

Suppose  $T_+ \neq 0$  and  $T_- \neq 0$ . Letting,  $t_0 = \inf\{t \mid \lambda(t, T_+) \geq 0\}$ , we have  $0 < t_0 < 1$  and

$$\lambda(t, T) = \begin{cases} \mu(t, T_+), & 0 < t < t_0 \\ \tilde{\mu}(1 - t, T_-), & t_0 \leq t < 1. \end{cases}$$



For  $r < t_0$ , we have

$$\begin{aligned} \tau(1_{[0, \mu(r, T)]}(|T|)T) &= \tau(1_{[-\mu(r, T), \mu(r, T)]}(T)T) = \tau(1_{[0, \mu(r, T)]}(T_+)T_+) - \tau(1_{[0, \mu(r, T)]}(T_-)T_-) \\ &= \int_{r'}^{t_0} \lambda(t, T) dt + \int_{t_0}^{1-r''} \lambda(t, T) dt, \end{aligned}$$

where

$$r' = \inf\{s \mid \mu(s, T_+) \leq \mu(r, T)\} \quad (16)$$

$$r'' = \inf\{s \mid \mu(s, T_-) \leq \mu(r, T)\}. \quad (17)$$

Since  $\mu(r, T_{\pm}) \leq \mu(r, T)$ , we have  $r', r'' \leq r$ . Thus, we have

$$\begin{aligned} \left| \tau(1_{[0, \mu(r, T)]}(|T|)T) - \int_r^{1-r} \lambda(t, T) dt \right| &= \left| \int_{r'}^r \lambda(t, T) dt + \int_{1-r}^{1-r''} \lambda(t, T) dt \right| \\ &\leq \int_{r'}^r \mu(t, T_+) dt + \int_{r''}^r \mu(t, T_-) dt \leq (r - r')\mu(r', T_+) + (r - r'')\mu(r'', T_-) \leq 2r\mu(r, T), \end{aligned}$$

where for the last inequality we used (16)–(17). This shows that the function (15) belongs to  $E$  and, thus, completes the proof.  $\square$

**Lemma 2.8.** *Let  $\varphi : \mathcal{E}(\mathcal{M}, \tau) \rightarrow \mathbb{C}$  be a trace. If  $T \in \mathcal{E}(\mathcal{M}, \tau)$  is self-adjoint and is such that  $\Psi\lambda(T) \in E$ , then  $\varphi(T) = 0$ .*

*Proof.* It follows from Theorem 2.7 that  $T \in [\mathcal{E}(\mathcal{M}, \tau), \mathcal{M}]$ . Since  $\varphi$  is a trace, it follows that  $\varphi(T) = 0$ .  $\square$

*Proof of Theorem 1.3.* For  $A \in \mathcal{S}(\mathcal{M}, \tau)$ , we have that  $A \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$  if and only if  $\log_+ \mu(A) \in E$ , and this is, in turn, equivalent to  $\log(1 + \mu(A)) \in E$ . Using the basic equalities (7)–(8), we easily see that for  $A, B \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$ , we have

$$\begin{aligned} \log(1 + \mu(A + B)) &\leq \log(1 + D_2\mu(A) + D_2\mu(B)) \leq \log((1 + D_2\mu(A))(1 + D_2\mu(B))) \\ \log(1 + \mu(AB)) &\leq \log(1 + D_2\mu(A)D_2\mu(B)) \leq \log((1 + D_2\mu(A))(1 + D_2\mu(B))), \end{aligned}$$

where  $(D_2f)(t) = f(t/2)$ . But since  $\log(1 + D_2\mu(A)) + \log(1 + D_2\mu(B)) \in E$ , these imply that  $A + B$  and  $AB$  belong to  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ . From this, one easily sees that  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$  is a  $*$ -subalgebra of  $\mathcal{S}(\mathcal{M}, \tau)$ .

It remains to show that  $\det_{\varphi}$  is multiplicative. Letting  $A, B \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$ , we will show (3). We may, without loss of generality, assume  $A, B \geq 0$ . Indeed, we have  $\mu(AB) = \mu(|A||B^*|)$ . Thus, if the assertion holds for positive operators, then we will have

$$\det_{\varphi}(AB) = \det_{\varphi}(|A||B^*|) = \det_{\varphi}(|A|)\det_{\varphi}(|B^*|) = \det_{\varphi}(A)\det_{\varphi}(B).$$

Suppose first that  $\log(A), \log(B) \in \mathcal{E}(\mathcal{M}, \tau)$ . Denote, for brevity,  $T = \log(A)$  and  $S = \log(B)$ . It follows from Lemma 2.2 that  $\log(|AB|) \in E$ .

Using Lemma 2.1 and replacing  $t$  with  $\frac{1}{2}t$ , for all  $t \in (0, \frac{1}{2})$ , we get

$$\left| \int_t^{1-t} (\log(\mu(u, e^T e^S)) - \lambda(u, T) - \lambda(u, S)) du \right| \leq 4t(\mu(\frac{t}{2}, T) + \mu(\frac{t}{2}, S)).$$

In particular, we have

$$\Psi(\log(\mu(e^T e^S)) - \lambda(T) - \lambda(S)) \in E.$$

It follows from Lemma 2.6 that

$$\Psi\left(\lambda(\log(|e^T e^S|)) - T - S\right) \in E.$$

Using Lemma 2.8, we conclude that

$$\varphi(\log(|e^T e^S|) - T - S) = 0.$$

This implies (3) for our  $A, B$ .

If  $B$  has a nonzero kernel, then so does  $AB$  and (3) holds.

Suppose now that  $\ker B$  is zero but  $\log_-(B) \notin E$ . Then, of course,  $\lim_{t \rightarrow 1} \mu(t, B) = 0$ . If  $\ker AB \neq \{0\}$ , then (3) holds, so suppose  $\ker AB = \{0\}$ . We have, from (8), for all  $t \in (0, \frac{1}{2})$ ,

$$\mu(1 - t, AB) \leq \mu(t, A)\mu(1 - 2t, B)$$

and, thus,

$$\log(\mu(1 - t, AB)) \leq \log(\mu(t, A)) + \log(\mu(1 - 2t, B)).$$

So, for sufficiently small  $t > 0$ ,

$$\begin{aligned} \log_- \mu(1 - t, AB) + \log_+ \mu(t, A) &\geq -\log \mu(1 - t, AB) + \log \mu(t, A) \\ &\geq -\log \mu(1 - 2t, B) = \log_- \mu(1 - 2t, B). \end{aligned}$$

Since the function  $t \mapsto \log_- \mu(1 - 2t, B)$  is not in  $E$ , while the function  $t \mapsto \log_+ \mu(t, A)$  does belong to  $E$ , we conclude that the function  $t \mapsto \log_- \mu(1 - t, AB)$  does not belong to  $E$ . Therefore, the function  $\log_-(\mu(AB))$  does not belong to  $E$  and both left- and right-hand sides of (3) are zero. This concludes the proof of (3) in the degenerate case.  $\square$

### 3. PROOF OF PROPOSITION 1.7 AND SOME EXAMPLES

**Lemma 3.1.** *Let  $m : \mathcal{E}_{\log}(\mathcal{M}, \tau) \rightarrow \mathbb{R}$  be multiplicative and order-preserving. Then for every  $T \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$ ,  $m(T)$  depends only on  $\mu(T)$ .*

*Proof.* We may without loss of generality assume  $m$  is not identically zero. Thus,  $m(1) = 1$ . By Theorem 1 of [1], every unitary element is a product of multiplicative commutators of unitaries (in fact, of symmetries) and it follows that  $m$  sends the entire unitary group of  $\mathcal{M}$  to 1. Thus, by employing the polar decomposition, we have

$$\forall T \in \mathcal{E}_{\log}(\mathcal{M}, \tau), \quad m(T) = m(|T|).$$

It, therefore, suffices to prove the assertion for positive operators.

Let  $0 \leq T, S \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$  be such that  $\mu(T) = \mu(S)$ . Set

$$T_\epsilon = \sum_{n \in \mathbb{Z}} (1 + \epsilon)^n 1_{((1+\epsilon)^n, (1+\epsilon)^{n+1})}(T), \quad S_\epsilon = \sum_{n \in \mathbb{Z}} (1 + \epsilon)^n 1_{((1+\epsilon)^n, (1+\epsilon)^{n+1})}(S).$$

For a given  $n$ , positive operators  $T_\epsilon$  and  $S_\epsilon$  have discrete spectrum and  $\mu(T_\epsilon) = \mu(S_\epsilon)$ . Since  $\mathcal{M}$  is a factor, one can choose a unitary operator  $U_\epsilon \in \mathcal{M}$  such that  $S_\epsilon = U_\epsilon T_\epsilon U_\epsilon^{-1}$ . Thus,

$$m(S_\epsilon) = m(U_\epsilon T_\epsilon U_\epsilon^{-1}) = m(U_\epsilon) m(T_\epsilon) m(U_\epsilon)^{-1} = m(T_\epsilon).$$

Clearly,

$$S_\epsilon \leq S \leq (1 + \epsilon) S_\epsilon, \quad T_\epsilon \leq T \leq (1 + \epsilon) T_\epsilon.$$

Since  $m$  is order preserving, it follows that

$$m(S) \leq m(1 + \epsilon) m(S_\epsilon) = m(1 + \epsilon) m(T_\epsilon) \leq m(1 + \epsilon) m(T).$$

Since  $m$  is order preserving, it follows that  $m(1 + \epsilon) \searrow 1$  as  $\epsilon \searrow 0$ . Passing  $\epsilon \rightarrow 0$ , we obtain  $m(S) \leq m(T)$ . Similarly,  $m(T) \leq m(S)$ . Thus,  $m(S) = m(T)$  and the proof is complete.  $\square$

*Proof of Proposition 1.7.* Since the map  $m$  is multiplicative and not identically zero, we must have  $m(1) = 1$ . By Lemma 3.1,  $m(T)$  depends only on  $\mu(T)$  for all  $T \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$ .

Let  $\mathcal{A}$  be any unital, diffuse, abelian von Neumann subalgebra of  $\mathcal{M}$ . As in the proof of Theorem 1.3,  $E$  is naturally identified with  $\mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$ . Given real-valued  $f \in E$ , let  $T \in \mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$  be the corresponding self-adjoint operator. Note that  $e^T$  is an invertible element of  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$  and, thus,  $m(e^T) > 0$ . We define

$$\varphi_0(f) = \log m(e^T). \quad (18)$$

We will show that  $\varphi_0$  is  $\mathbb{R}$ -linear. First, given  $f_1, f_2 \in E$  and the corresponding self-adjoint  $T_1, T_2 \in \mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$ , since  $T_1$  and  $T_2$  commute, we have

$$\varphi_0(f_1 + f_2) = \log m(e^{T_1+T_2}) = \log m(e^{T_1}e^{T_2}) = \log (m(e^{T_1})m(e^{T_2})) = \varphi_0(f_1) + \varphi_0(f_2),$$

i.e.,  $\varphi_0$  preserves addition. From this, we easily see that  $\varphi_0(rf) = r\varphi_0(f)$  for every rational number  $r$  and real-valued  $f \in E$ . This last fact is, of course, equivalent to

$$m(e^{rT}) = m(e^T)^r \quad (19)$$

for every self-adjoint  $T \in \mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$  and every rational number  $r$ . When  $T \geq 0$ , using the order-preserving property of  $m$ , we obtain from this that (19) holds for every  $r \in \mathbb{R}$ , and similarly when  $T \leq 0$ . For arbitrary self-adjoint  $T \in \mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$ , writing  $T = T_+ - T_-$  for  $T_+$  and  $T_-$  positive elements of  $\mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$ , in the usual way, we get, for all  $r \in \mathbb{R}$ ,

$$\begin{aligned} m(e^{rT}) &= m(e^{rT_+ + (-r)T_-}) = m(e^{rT_+})m(e^{(-r)T_-}) = m(e^{T_+})^r m(e^{T_-})^{-r} \\ &= (m(e^{T_+}e^{-T_-}))^r = m(e^T)^r. \end{aligned}$$

Thus (19) holds for all self-adjoint  $T$  and all  $r \in \mathbb{R}$ , and it follows that  $\varphi_0(rf) = r\varphi_0(f)$  for all real-valued  $f \in E$  and all  $r \in \mathbb{R}$ . Thus, we have defined an  $\mathbb{R}$ -linear functional  $\varphi_0$  on the space of real-valued elements of  $E$ . Complexification extends  $\varphi_0$  to a  $\mathbb{C}$ -linear functional on  $E$ .

We now observe that  $\varphi_0$  is rearrangement-invariant. If  $f \in E$  and  $f \geq 0$  and if  $T \in \mathcal{S}(\mathcal{A}, \tau|_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$  is the corresponding element, then  $\mu(e^T) = e^{f^*}$ , where  $f^*$  is the nondecreasing rearrangement of  $f$ . Since  $m(e^T)$  depends only on  $\mu(e^T)$ , we see that  $\varphi_0(f) = \varphi_0(f^*)$  and, thus,  $\varphi_0$  is rearrangement-invariant.

By Theorem 1.1, there is a unique trace  $\varphi$  on  $\mathcal{E}(\mathcal{M}, \tau)$  such that  $\varphi(T) = \varphi_0(\mu(T))$  whenever  $T \in \mathcal{E}(\mathcal{M}, \tau)$  is positive. Suppose  $X$  is an invertible element of  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$  and let us observe that  $m(X) = \det_{\varphi}(X)$ . Since  $m(X) = m(|X|)$  and likewise for  $\det_{\varphi}$ , we may without loss of generality assume  $X \geq 0$ . Thus, there is self-adjoint  $T = \log(X) \in \mathcal{E}(\mathcal{M}, \tau)$  such that  $X = e^T$ . Thus, by (18), we have

$$m(X) = e^{\varphi_0(\lambda(T))} = e^{\varphi(T)} = \det_{\varphi}(X),$$

as required.  $\square$

The following shows that Proposition 1.7 cannot be improved to obtain  $m = \det_{\varphi}$  on all of  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ .

**Proposition 3.2.** *Let  $E$  be a symmetric function space. Consider strictly larger symmetric function space  $F$ . If  $\psi$  is an arbitrary positive trace on  $\mathcal{F}(\mathcal{M}, \tau)$ , then*

$$\det_{\psi}|_{\mathcal{E}_{\log}(\mathcal{M}, \tau)} \neq \det_{\varphi}$$

*for each positive trace  $\varphi$  on  $\mathcal{E}(\mathcal{M}, \tau)$ .*

*Proof.* To see this, fix  $0 \leq T \in \mathcal{F}(\mathcal{M}, \tau)$  such that  $T \notin \mathcal{E}(\mathcal{M}, \tau)$ . Take  $X = e^{-T}$ . Then  $X$  is bounded, so belongs to  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ . Moreover,  $X^{-1} = e^T$  belongs to  $\mathcal{F}_{\log}(\mathcal{M}, \tau)$ , but  $X$  is not invertible in  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ . Thus, we have

$$\det_{\psi}(X) = e^{-\psi(T)} \neq 0 = \det_{\varphi}(X).$$

□

See Remark 1.6 for the relevance of the following example.

**Example 3.3.** We give examples of a nonzero trace  $\varphi$  on a bimodule  $\mathcal{E}(\mathcal{M}, \tau)$  and  $T \in \mathcal{E}(\mathcal{M}, \tau)$  such that  $\varphi \neq 0$  but

$$\det_{\varphi}(T) \neq \lim_{\epsilon \rightarrow 0^+} \det_{\varphi}(|T| + \epsilon). \quad (20)$$

Let  $\psi$  be an increasing, continuous, concave function on the interval  $[0, 1]$  satisfying

$$\lim_{t \rightarrow 0} \frac{\psi(2t)}{\psi(t)} = 1.$$

For example, take  $\psi(t) = \frac{1}{2 - \log(t)}$ . Let  $E = M_{\psi}$  be the Marcinkiewicz space

$$E = \left\{ f \in S(0, 1) \mid \sup_{0 < t < 1} \frac{1}{\psi(t)} \int_0^t f^*(s) ds < \infty \right\},$$

where  $f^*$  is the decreasing rearrangement of  $|f|$ . Let  $\mathcal{E}(\mathcal{M}, \tau)$  be the corresponding  $\mathcal{M}$ -bimodule. By Example 2.5(ii) of [3], there is a positive, rearrangement-invariant, linear functional  $\varphi$  on  $E$  that vanishes on  $E \cap L_{\infty}$ , but satisfies  $\varphi(\psi') = 1$ . For  $f \in E$  with  $f \geq 0$ ,  $\varphi(f)$  is realized as a particular sort of generalized limit as  $t \rightarrow 0$  of  $\frac{1}{\psi(t)} \int_0^t f^*(s) ds$ . Let  $\varphi$  denote also the trace on  $\mathcal{E}(\mathcal{M}, \tau)$ , according to Theorem 1.1. Thus, we have  $\det_{\varphi}(T) = 1$  whenever  $T \in \mathcal{M}$  is bounded and has bounded inverse. Consequently, if  $T \in \mathcal{M}$  fails to be invertible in  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ , for example, because it has a nonzero kernel, then, by Definition 1.2,  $\det_{\varphi}(T) = 0$ , but the right-hand-side of (20) is equal to 1.

The examples considered hitherto involved non-invertible elements of  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ . However, (20) can also fail when  $T$  is invertible in  $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ . For example, take  $T \geq 0$  such that  $\mu(T)(t) = \exp(-\psi'(1 - t))$ . In particular,  $T$  is bounded. Then  $\det_{\varphi}(T) = e^{-1}$  but again the right-hand-side of (20) is equal to 1.

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